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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

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International Workshop on Stochastic Filtering Theory

Chapel Hill, N.C., June 26–28, 1994

G. Kallianpur, Organizer

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13. ABSTRACT *(Maximum 200 words)*

A total of 39 participants registered and attended this workshop, held on the campus of the University of North Carolina in Chapel Hill, to review the most recent advances in major new areas of filtering theory. They came from 17 foreign countries and the United States. There were major talks by 20 distinguished mathematicians and shorter talks by three young researchers in the field.

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Report on International Workshop on Stochastic Filtering Theory, June 26–28, 1994

A total of 39 participants registered and attended this workshop, held on the campus of the University of North Carolina in Chapel Hill, to review the most recent advances in major new areas of filtering theory. They came from 17 foreign countries and the United States. There were major talks by 20 distinguished mathematicians and shorter talks by three young researchers in the field.

I give below a very rough classification of the papers presented:

Theoretical aspects of nonlinear filtering theory:	14
Risk sensitive control problems:	2
Asymptotic methods:	4
Numerical and approximation methods	3.

On the last day, Professor F. LeGland gave a special lecture on "Numerical methods in optimal nonlinear filtering." Professor E. Platen also gave a brief informal talk on "Higher order approximate nonlinear filters." The two talks were followed by a lively discussion.

As our technical report, we attach the following:

1. Complete schedule for the three days of the Workshop.
2. List of participants in attendance, and their affiliations.
3. Abstracts for the 23 talks presented at the Workshop.
4. Outline of the survey on Numerical Methods in Optimal Nonlinear Filtering.

G. Kalliampur.

G. Kallianpur, Principal Investigator

Sponsored by the U.S. Army Research Office, Mathematical Sciences Division, through grant DAAH04-94-G-0077.

gk/lt

Stochastic Filtering Workshop

(310 Peabody Hall, University of North Carolina)

Registration: Saturday, June 25, 8—11 p.m., Carolina Inn Lobby

Sunday, June 26

Morning

8:45 a.m.		Welcoming Remarks
9:00 – 9:50	E. Pardoux	"Some asymptotic results in nonlinear filtering"
9:55 – 10:45	B. Grigelionis	"Infinite dimensional integrators in nonlinear filtering"
11:00–11:50	A. Bensoussan (w/R.J. Elliott)	"On risk sensitive stochastic control problems with partial information"
11:55–12:45	B.L. Rozovskii (w/R. Mikulevicius)	"Cameron Martin development as a numerical algorithm for nonlinear filtering"
12:45 – 2:30	Lunch	

Afternoon

2:30 – 3:20	E. Platen	"Higher order and implicit Markov chain filters"
3:25 – 4:15	J.S. Baras	"Stochastic filtering theory and nonlinear robust control"
4:30 – 5:20	E. Mayer-Wolf	"Conditional densities in filtering and other problems"
5:30 – 6:00	R. Leland	"A new formula for the log-likelihood gradient for continuous time stochastic systems"

Monday, June 27

Morning

9:00 – 9:50	H. Kunita	"The asymptotic behaviors of solutions of Zakai equations"
9:55 – 10:45	J.M.C. Clark	"Questions of prior dependence and approximation for discrete-parameter Markov filters"
11:00–11:50	F. Le Gland	"Nonlinear filtering with perfect observations"
11:55–12:45	R. Mikulevicius (w/B. Rozovskii)	"On Hölder continuity of solutions to Zakai equations"
12:45 – 2:30	Lunch	

Afternoon

2:30 – 3:20	R.L. Karandikar(w/A. Bhatt & G. Kallianpur)	"Uniqueness of solutions of the Zakai equations"
3:25 – 4:15	H. Korezlioglu	"Approximation of filters by random sampling"
4:30 – 5:00	R. J. Elliott (w/M. James)	"Risk sensitive control of a partially observed Markov chain"
5:10 – 5:40	A. Budhiraja (w/G. Kallianpur)	"A system of integral equations in nonlinear filtering"
8:00	Dinner	

Tuesday, June 28

Morning

9:00 – 9:50	D.L. Ocone	"Asymptotic dependence of optimal filters on initial priors"
9:55 – 10:45	A. Bagchi	"Estimation of nonlinear boundary value processes"
11:00–11:50	W.J. Runggaldier (w/L. Stettner)	"Large time behavior of the filter corresponding to discrete time partially observed stochastic control problems"
11:55–12:45	V. Beneš	"Nonlinear filtering in action: Solvable examples from optimal control"
12:45 – 2:30	Lunch	

Afternoon

2:30 – 3:20	O. Enchev	"Signals, noise and filtering on manifolds"
3:25 – 3:55	P. Florchinger	"Filtering with a discontinuous observation on manifold"
4:00 – 4:20	D. Crisan	"Explicit formulae for the conditional densities for some finite dimensional filters"

4:30 – 5:00	F. LeGland gives a brief survey of discrete approximations to solutions of the Zakai equation"
5:05 – 5:35	Round Table Discussion

Stochastic Filtering Theory Workshop

University of North Carolina – Chapel Hill
June 25 – 28, 1994

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Estimation of Nonlinear Boundary Value Processes

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Abstract

Boundary value processes are stochastic processes satisfying ordinary or partial differential equations with, possibly random, boundary conditions. For linear equations, existence of such processes and the associated estimation problems have been extensively studied in the literature [1]. The mathematical formulation of such boundary-value problems for nonlinear equations is already a difficult one. This is due to the nature of Itô integrals, which are defined forward in time. We study this problem when the input noise is modelled directly as a finitely-additive white noise [2], instead of the usual approach where one works with the Brownian motion [2]. We consider a semilinear two-point boundary value problem

$$\dot{x}(t) = A(t)x(t) + f(x(t)) + n(t)$$

$$F_0x(0) + F_1x(1) = F$$

where A, F_0, F_1 are $m \times m$ matrices, F is an m-dimensional random vector, $n(\cdot)$ is a finitely additive Gaussian white noise, and f is bounded and Lipschitz continuous. We show that the solution process $\{x(\cdot)\}$ exists and is unique. We then consider the associated estimation problem. Let the observation process be given by

$$y(t) = x(t) + n_0(t)$$

where $n_0(\cdot)$ is another Gaussian white noise. We solve the smoothing problem of estimating $x(t)$, for fixed t, based on observing $y(s); 0 \leq s \leq 1$. Estimation problem for a different type of boundary value processes where the process satisfies a nonlinear elliptic partial differential equation has been studied in [3].

References

[1] A. Bagchi, "Boundary value processes: estimation and identification", *Computers and Mathematics with Applications, 19*, pp. 9-19, 1990.

[2] G. Kallianpur and R.L. Karandikar, *White Noise Theory of Filtering, Prediction and Smoothing*, Gordon and Breach, London, 1988.

[3] S.-I. Aihara and A. Bagchi, "Nonlinear smoothing for random fields", *Stochastic Processes and Their Applications* (accepted for publication).

Stochastic Filtering Theory, Risk-Sensitive Stochastic Control and Nonlinear Robust Control

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Abstract

We present recent advances in output robust control of nonlinear systems, using stochastic control and stochastic filtering methods. We first establish the equivalence of three seemingly unrelated problems: (a) The so called four-block formulation of the nonlinear output feedback robust control problem; (b) A partially observed differential game with the same dynamics as (a); and (c) A risk-sensitive, partially observed stochastic control problem. Working first in (c) we obtain naturally an "information state" for the problem which summarizes the past input and output histories as needed by the controller. We also present the dynamical equation satisfied by the "information state". We also present the Dynamic Programming equation that the value function of (c) satisfies. In the formulation of (c) from (b) we introduced a "small" noise in the system dynamics and in the observation equation.

We now take the limit as the noise vanishes. We then obtain a Hamilton-Jacobi-Isaacs pde characterizing the value of the differential game, and another pde characterizing the "information state" of the differential game. We then show how the differrential game solution provides the general solution to the output feedback robust control problem for nonlinear systems. The controller has two parts: (i) A dynamic "observer" that is essentially the dynamical system describing the evolution of the information state of the associated differential game (see problem (b) above); (ii) A memoryless feedback control as a function of the information state, obtained from the off-line solution of the H-J-I p.d.e. of the associated differential game.

We also discuss the computational complexity of the general solution and approximate solutions provided by certainty-equivalence controllers. We present computational results on highly nonlinear, low order (≤ 2) nonlinear control problems, which clearly illustrate that the method produces stabilizing controllers. We show that for robust stabilization by output feedback control the H-J-I equation leads to a Partial Differential Inequality in the sense of dissipative systems. We finally present a framework for studying the tradeoffs between model accuracy, feedback control performance and complexity.

Nonlinear Filtering in Action: Solvable Examples from Optimal Control.

Vaclad Beneš

Abstract: Some simple problems of control with incomplete information are compared in point of structure, optimal law, ease of solution, and relation to Mortensen's equation.

A FINITE DIMENSIONAL RISK SENSITIVE CONTROL PROBLEM

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Abstract. A partially observed stochastic control problem with exponential running cost is considered. The dynamics are linear and the running cost quadratic, though the control may enter nonlinearly. Explicit solutions are found to a modified Zakai equation and a backward adjoint equation. This enables the problem to be expressed in terms of observable finite dimensional dynamics and a separation principle applied.

Uniqueness and robustness of solution of measure valued equations of nonlinear filtering [1]

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Center for Stochastic Processes
University of North Carolina at Chapel Hill

and

Rajeeva L. Karandikar
Indian Statistical Institute, New Delhi

Abstract

We consider the Zakai equation for the unnormalised conditional distribution σ when the signal process X takes values in a complete separable metric space E and when h is a continuous, possibly unbounded function on E. It is assumed that X is a Markov process which is characterized via a martingale problem for an operator A_0. Uniqueness of solution for the measure valued Zakai and FKK equation is proved when the test functions belong to the domain of A_0. It is also shown that conditional distributions are robust.

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Key words and phrases: Nonlinear filtering, Zakai equation, Martingale problem, robustness

A System of Integral Equations in Nonlinear filtering.

A. Budhiraja and G. Kallianpur

In this work we study multiple integral representations in problems of nonlinear filtering. The first part of the talk focuses on multiple Stratonovich integrals. It is shown that the unnormalized conditional expectation in the filtering problem can be expressed as an infinite sum of such multiple integrals. A mean square bound, for the error on truncating the infinite series, is obtained. A discrete scheme for approximating the conditional expectation is indicated.

In the second part of the talk, multiple Wiener integral representations for the conditional density are studied and a system of integral equations for the kernels is given. It is shown that the system admits a unique solution. This also leads to a system of partial differential equations satisfied by the coefficients of the kernels in the multiple integral representation for the conditional density.

Questions of Prior Dependence and Approximation For Discrete-Parameter Markov Filters

J.M.C. Clark

Abstract

In this talk the asymptotic properties of discrete-parameter nonlinear filters and their approximations will be considered. One question to be addressed is whether a filter 'forgets' its prior distribution. Theorems on this in the continuous-parameter case have been produced by Kunita and Ocone in which conditional distributions from different 'priors' converge weakly to each other; partial results will be discussed for the discrete-parameter case in which the conditional distributions converge in total variation. A second topic to be considered concerns the long-term behavior of a hierarchy of approximate filters that converge to an ideal filter; here the question is whether the approximate filters remain close to the ideal filter over long periods of time.

Explicit Formulae for the Conditional Densities For Some Finite Dimensional Filter

D. Crisan, University of Edinburg

The explicit form for the unnormalised measure for the Beneš filter and for the normalised measure for the linear filter are derived. These forms are simpler than the conventional ones and in the second case we obtain the probability measure without solving the Ricatti equation and the SDE to find out the covariance matrix and the mean. The method involves the explicit expression for the quantity:

$$E\left[\exp\left(F(W_t) + \int_0^t W_s^* f(s)\,ds - \frac{1}{2}\int_0^t |AW_s|^2\,ds\right) \mid W_t = a\right]$$

where $\{(W_t);\ t \geq 0\}$ is a d-dimensional standard Brownian motion, A is a $d \times d$ real matrix, F: $\mathbb{R}^d \rightarrow \mathbb{R}$ and f: $[0,t] \rightarrow \mathbb{R}^d$ is square integrable.

Risk-Sensitive and Risk-Neutral Control for Continuous-Time Hidden Markov Models

M.R. James*   R.J. Elliott†

Abstract

In this paper the optimal control of a continuous time hidden Markov model is discussed. The risk sensitive problem involves a cost function which has an exponential form and a risk parameter, and is solved by defining an appropriate information state and dynamic programming. As the risk parameter tends to zero, the classical risk neutral optimal control problem is recovered. The limits are proved using viscosity solution methods.

Signals, Noise and Filtering on Manifolds

O.B. Enchev
Boston University

Abstract: In flat spaces, the most commonly used filtering scheme has the form:

observation = signal + noise.

In spaces with curvature the "+" above isn't relevant and one has to use a different scheme, which takes into account the fact that the state-space is curved. I will propose one such scheme. It is based on a method, developed jointly with D.W. Stroock (MIT). The key instrument is what we call "diffusion of smooth curves", described via C^∞-valued SDE's (we treat such equations in ultra-strong sense; not in generalized sense after integration against test functions — this was possible with a trick due to D.W. Stroock). Our C^∞-valued SDE may be regarded as a stochastic analog of Cartan's first fundamental form.

We also obtain an analog of Cameron Martin's theorem and integration by parts in Wiener's path space on compact Riemannian manifold.

Filtering with a discontinuous observation on manifold.

Patrick Florchinger
Université de Metz

Abstract: We consider a random signal (a vectorial Markovian process) partially observed on a symmetric Riemannian space. The observation is a càdlàg stochastic process given by a multiplicative perturbation of the signal. We prove that the filter is absolutely continuous with respect to the Lebesgue measure. The main tool used in this paper is the stochastic calculus of variations (Malliavin calculus).

INFINITE DIMENSIONAL INTEGRATORS IN NONLINEAR FILTERING

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Abstract

Let $(\mathcal{X}, \mathcal{B})$ be a measurable space with the countably generated σ-algebra $\mathcal{B}$, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis, $\bar{\mathbb{F}}$ be the subfiltration of $\mathbb{F}$ and $\{\theta_t, t \geq 0\}$ be a $\mathbb{F}$-adapted $\mathcal{X}$-valued stochastic process. Extending the well known method of M.Fujisaki, H.Kunita and G.Kallianpur, we shall discuss the nonlinear filtering equation for the aposteriori distributions $P\{\theta_t \in B \mid \bar{\mathcal{F}}_t\}$, $t \geq 0$, $B \in \mathcal{B}$, basing on the representation properties of the $(P, \bar{\mathbb{F}})$-local martingales as the sums of stochastic integrals with respect to the defined continuous quasicomplete locally convex topological vector space valued local martingale and the compensated point measure on a Blackwell space, as well as the transformation properties of changes of filtrations. It will be also presented the linear stochastic equation (Zakai's equation) for the unnormalized aposteriori distribution in this generality.

Approximation of Filters by Random Sampling

Hayri Korezlioglu

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The following system of state and observation equations is considered

$$\begin{aligned} x(t) &= x_0 + \int_O^t f(x_s)ds + \int_O^t g(x_s)dB_s \\ y(t) &= \int_O^t h(x_s)ds + W_t \end{aligned}$$

where B and W are two real valued independent Brownian motions and independent of x_0, and f, g and h are real functions having Lipschitz and linear growth properties, so that the system has a unique continuous Markovian solution. The approximation problem by periodic sampling was considered in [1]. Here the sampling points are supposed to form a sequence of stopping times depending on the observation process y. Similarly to [1] an approximation of the filter at sampling points is proposed and the convergence of the approximate filter to the true one is considered. Finally, the feasibility of the approach is dicussed. The results obtained in this work can be extended to a more general multidimensional case with functions f, g and h also depending on y as in [1].

[1] H. Korezlioglu and W.J. Runggaldier:*Filtering for nonlinear systems driven by nonwhite noises: an approximation scheme*, Stochastics and Stochastics Reports, Vol. 44, pp. 65-102.

The asymptotic behaviors of solutions of Zakai equations

Hiroshi KUNITA

We consider a stochastic partial differential equation on $\mathbf{R}^d$, called a Zakai equation.

$$\begin{aligned} u(x,t) &= f_0(x) + \int_0^t Lu(x,r)dr + \sum_{k=1}^n \int_0^t Y_k u(x,r)dW^k(r) \\ &+ \sum_{k=1}^n \int_0^t h_k(x)u(x,r)dW^k(r). \end{aligned}$$

Here L (Y_k) is a second order (a first order) linear partial differential operator, and $(W^1(t),\dots,W^n(t))$ is an n-dimensional standard Brownian motion. We discuss the asymptotic behavior of the solution $u(x,t)$ as t tends to infinity, comparing this with the asymptotic property of the solution of the equation excluding the terms of random perturbations, i.e., the solution of the partial differential equation

$$\frac{\partial \overline{u}}{\partial t} = L\overline{u}, \quad u|_{t=0} = f_0.$$

The asymptotic behavior of the solution $\overline{u}(x,t)$ depends on the recurrence or the transience of the diffusion process with infinitesimal generator L, as is well known. We will show that the asymptotic behavior of the solution $u(x,t)$ also depends on whether the diffusion process with the infinitesimal generator L is transient, null recurrent or positive recurrent.

Assuming that the initial function $f_0(x)$ satisfies $\lim_{|x|\to\infty} f_0(x) = 0$, the solution $u(x,t)$ converges to 0 in L^p for any $p \geq 1$ if the associated diffusion process with generator L is transient or null recurrent. However, if the associated diffusion process is positive recurrent with invariant probability Λ, the situation is complicated. Results are summarized as follows under Conditions (C.1) and (C.2) to be stated precisely in the lecture.

(1) Assume $h_k = 0$, $k = 1,\dots,n$. Then the time average $T^{-1}\int_0^T u(x,t)dt$ of $u(x,t)$ converges to $\Lambda(f_0) \equiv \int f_0(x)\Lambda(dx)$ a.s. and in L^p $(p \geq 1)$ as $t \to \infty$ for all x. However $u(x,t)$ does not converge to $\Lambda(f_0)$ in L^p $(p \geq 1)$ as $t \to \infty$ in general. The convergence takes place if and only if the initial function f_0 satisfies $\Lambda(Y_k f_0) = 0$ for any $k = 1,\dots,n$.

(2) Assume $h_k \neq 0$ for some k. Then $u(x,t)$ converges to 0 a.s. but does not converge to 0 in L^p $(p \geq 1)$ if $f_0 \neq 0$.

Nonlinear Filtering with Perfect Observations

Marc Joannides (INRIA Sophia–Antipolis)
François LeGland (IRISA/INRIA Rennes)

Abstract : We are interested in nonlinear filtering problems, where the observation noise has a singular covariance matrix. There are indeed numerous situations of practical interest, where some perfect noise–free information is available about the unknown state. Another motivation is that a better understanding of the singular case should help designing robust and efficient numerical procedures in the important case of a small observation noise. To be specific, we consider an hypoelliptic diffusion process $\{X_t\,,\, t \geq 0\}$ in $\mathbf{R}^m$, with noise–free d–dimensional discrete time observations

$$z_k = h(X_{t_k}) \ .$$

What makes this problem singular is that the state X_{t_k} is known *exactly* to belong to the level set $M_k = M(z_k)$, where for all $z \in \mathbf{R}^d$

$$M(z) = h^{-1}(z) = \{x \in \mathbf{R}^m \ : \ h(x) = z\} \ .$$

Therefore, the conditional probability distribution $\mu_k(dx)$ of the state X_{t_k} given observations $\{z_0, \cdots, z_k\}$, is supported by the set M_k, i.e. can not in general have a density w.r.t. the Lebesgue measure on $\mathbf{R}^m$.

The first step is to reduce the original problem as follows : Let X be a r.v. in $\mathbf{R}^m$ with probability distribution $\mu(dx) = p(x)\,dx$ and continuous density $p(x)$, and let $Z = h(X)$. What is the conditional probability distribution of the r.v. X given $[Z = z]$?

If $z \in \mathbf{R}^d$ is a regular value of the mapping h, the solution is provided by the well–known area or co–area formula depending on whether $m \leq d$ or $m > d$, see Federer (1969) or Evans and Gariepy (1992). However, we are interested in studying also the case of singular values $z \in \mathbf{R}^d$.

For this purpose, we introduce the following perturbation approach : From Takeuchi and Akashi (1985), we know that for any test function ϕ defined in $\mathbf{R}^m$

$$\mathbf{E}[\phi(X) \mid Z^\varepsilon] \longrightarrow \mathbf{E}[\phi(X) \mid Z]$$

in probability as $\varepsilon \downarrow 0$, where $Z^\varepsilon = Z + V^\varepsilon = h(X) + V^\varepsilon$, and V^ε is a d–dimensional Gaussian r.v. with covariance matrix $\varepsilon\, I_d$, independent of the r.v. X. The problem reduces then to studying the asymptotic behaviour as $\varepsilon \downarrow 0$ of some Laplace integrals. In the case of a regular value $z \in \mathbf{R}^d$, we adapt the result of Hwang (1980) to our situation, and hopefully recover the results provided by the direct approach. In the case of a singular value $z \in \mathbf{R}^d$, we obtain some partial results when $m \leq d$, using ideas contained in Ellis and Rosen (1982). We illustrate the results with some examples.

We conclude with another application of the asymptotics of Laplace integrals : We consider the parameter estimation problem in a diffusion process with small noise, when the standard identifiability assumption does not hold, and we obtain a consistency result for the Bayesian estimate, which extends a recent result of Kutoyants (1992).

A Log-Likelihood Gradient for Continuous Time Stochastic Systems Using Finitely Additive White Noise Theory

Robert Leland

University of Alabama

Using a finitely additive white noise approach, an explicit expression is derived for the gradient of the log-likelihood ratio for estimating parameters of a continuous time linear stochastic system from noisy observations:

$$\frac{dx(t)}{dt} = Ax(t) + Fw(t)$$

$$y(t) = Cx(t) + Gv(t)$$

where w(t) and v(t) are finitely additive Gaussian white noise processes. The gradient formula includes the smoother estimates of the state vector, and derivatives of only the system matrices, and not the estimates or error covariances. A scheme to calculate the log-likelihood gradient without solving any Riccati equations is described when only A and the initial covariance are functions of the unknown parameter.

Conditional Densities in Filtering and Other Problems

Eddy Mayer-Wolf

Abstract

In filtering problems one is often interested in the question whether there exists a density for the conditional law of the state. This issue will be addressed in the following general form: given that a functional U of a process x_t possesses a density, what must be assumed for the same to remain true when U's law is conditional on a second process y_t?

The answer involves a certain entropy continuity property of y's law conditioned on x, and can be considered to be a qualitative version of Bayes rule.

Concrete examples which illustrate these results will be discussed as well.

On Hölder Continuity of Solution to Zakai Equations

R. Mikulevicius
University of Southern California, Los Angeles

We consider Zakai type nondegenerated equations with bounded coefficients:

$$\begin{cases} du = (a^{ij}u_{x_ix_j} + b^iu_{x_i} + cu + f)\,dt + (hu+g)dw_t \\ u(0,\cdot) = 0, \quad (t,x) \in [0,1] \times \mathbb{R}^d. \end{cases}$$

We look at the solution as a function of (t,x) with values in $E = L_p(\Omega,\mathcal{F},P)$. If a, b, c, f belong to the Hölder space C^α and h, g belong to the Hölder space $C^{1+\alpha}$ $(\alpha \in (o,1))$, we find a solution $u \in C^{2+\alpha}(E)$ such that $|u|_{2+\alpha} \leq \text{const}(|f|_\alpha + |g|_{1+\alpha})$.

Asymptotic Dependence of Optimal Filters on Initial Priors

D. Ocone

ABSTRACT

Our talk surveys known results for the problem of "forgetting of initial conditions" in nonlinear filtering of Markov processes. For a Markov process (X,Y), the condition distribution Π_t, of X_t given $Y_t^0 = (Y_s,\ 0 \le s \le t)$ is computed by a Bayes rule formula $\Pi_t = \sqcap_t(\nu, Y_t^0)$ taking two inputs, the observation path and the initial law ν of (X_0, Y_0), which determines the prior law on (X,Y) by virtue of the Markov property. Roughly speaking, we say that the filter forgets initial conditions if $\Pi_t - \overline{\Pi}_t \to 0$ in some sense as $t \to \infty$ where $\overline{\Pi}_t = \sqcap_t(\overline{\nu},\ Y_t^0)$ represents a filter computed with an erroneous initial law $\overline{\nu} \ne \nu$. We discuss the following main points a) (J.M.C. Clark) the relative entropy between Π_t and $\overline{\Pi}_t$ as a positive supermartingale, b) almost sure exponential convergence of linear filters with incorrect priors, c) L^2-convergence of $\Pi_t(\vartheta) - \overline{\Pi}_t(\vartheta)$ to 0 for ergodic signals in additive white noise. The talk presents joint work with E. Pardoux.

Filtering of a Partially Observed Diffusion with High Signal-to-Noise Ratio

E. Pardoux

(joint work with A. Gégout-Petit)

J. Picard was the first one to prove rigorous results about approximate finite dimensional nonlinear filters for one dimensional diffusion observed in small noise. We consider the more general filtering problem:

$$X_t^1 = x^1 + \int_0^t f_1(X_s^1, X_s^2)\, ds + B_t^1$$

$$X_t^2 = x^2 + \int_0^t f_2(X_s^1, X_s^2)\, ds + B_t^2$$

$$Y_t = \int_0^t h(X_s^1)\, ds + \epsilon W_t,$$

where B^1, B^2 and W are mutually independent standard Brownian motions, f_1 and f_2 are Lipschitz functions, and $h \in C^1(\mathbb{R})$ and satisfies: $0 < a \le h'(x) \le b$, $x \in \mathbb{R}$.

We propose an approximate filter which consists of a one dimensional Picard-type filter for X_t^1, and an approximate conditional law for X_t^2, which solves an SPDE with one dimensional space variable, and give an estimate of the difference between our approximate and the optimal filter.

Higher Order and Implicit Markov Chain Filters

Eckhard Platen

Australian National University

Institute of Advanced Studies, Canberra

__Abstract__: The talk considers some aspects of the construction of hidden Markov chain filters for discrete time numerical methods in stochastic differential equations. Strong discrete time schemes lead to high order filters. The derivation of implicit filters allows a considerable increase of the stability of corresponding filtering algorithms.

B.L. Rozovskii[1]
University of Southern California, Los Angeles

Cameron-Martin Development as a Numerical Algorithm for Nonlinear Filtering

We introduce a modification of the Wiener chaos expansion to give an explicit solution to the Zakai equation. We show that this representation is of the form

$$u(t,x) = \sum_{\alpha} \varphi_{\alpha}(t,x) h^{\alpha}(y)$$

where functions φ_{α} depend only on the coefficients of the Zakai equation and functions h^{α} are determined only by the observations. We discuss applications of this formula to the numerical solution of the Zakai equation.

[1]This talk is based on joint works with R. Mikulevicius and S. Lototsky

Large time behaviour of the filter corresponding to discrete time partially observed stochastic control problems

Wolfgang J. Runggaldier and Lukasz Stettner

Investigating filtering problems is useful for various purposes, among which also the study of stochastic control problems under partial observation of the state. In the case of infinite horizon problems with ergodic cost, the long time behaviour of the filter becomes then important. In fact, if one considers controls that are functions of the current filter, the filter process itself becomes Markov and, under certain assumptions, the ergodic cost function can then be expressed as an integral with respect to an invariant measure of the filtering process. It follows that, when studying e.g. approximations for ergodic cost problems, it is important to have results on approximations, convergence, as well as uniqueness of invariant measures of the filter corresponding to partial observation control problems.

While for uncontrolled models there exist results characterizing the ergodic properties of the filter process in terms of the underlying state process, for controlled models such results are much more difficult. We discuss two possibilities to obtain a unique invariant measure for discrete-time controlled filtering processes. One is given by the case when the controlled filter process admits an embedded i.i.d. process (periodic return, with bounded average time, of the filter to a same measure). In this case, for which an example is shown, one can use the Law of Large Numbers and obtain not only existence and uniqueness of an invariant measure, but also its representation and this is particularly useful to obtain convergence results for approximations. A second possibility is given by the so-called "mixed observation" case , where one has perfect observation inside a given recurrent subset of the state space and partial observation outside.

Numerical Methods in Optimal Nonlinear Filtering

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Algorithms that work today

- extensive simulations (mainly 1d) on a variety of examples

  Software (ZPB)

  robust and reliable, i.e., can handle

  wrong initial condition

  wrong parameters

  small observation noise

- applications to some real-world problem

Observations $\{Z_t,\ t \geq 0\}$ about some unobserved state process $\{X_t,\ t \geq 0\}$

$$Z_t = h(X_t) + v_t$$

$$(\text{or: } dY_t = h(X_t)dt + dV_t)$$

Want to estimate X_t from $\mathcal{Y}_t = \sigma(Y_s,\ 0 \leq s \leq t)$, i.e., to compute

$$\langle p_t, f \rangle = E^{\dagger}[f(X_t)\, Z_t \mid \mathcal{Y}_t]$$

with

$$Z_t = \frac{dP}{dp^{\dagger}} \mid \mathcal{F}_t = \exp\left\{ \int_0^t h(X_s)dY_s - \tfrac{1}{2} \int_0^t |h(X_s)|^2 ds \right\}$$

Bayesian point of view: model for $\{X_t,\ t \geq 0\}$

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \rho(X_t dV_t$$

$$Z_t = h(X_t) + v_t$$

Monte-Carlo method

$$\frac{1}{N}\sum_{p=1}^{N} f(X_t^p)\, Z_t^p \;\rightarrow\; \langle p_t, f\rangle$$

as $N \rightarrow \infty$, but

- very delicate to implement [FℓG, CDC'84, Las Vegas]
- less reliable than other methods

Time-evolution for $\{p_t,\ t \geq 0\}$

$$dp_t \;=\; L^* p_t\, dt + \sum_{k=1}^{d} B_k^* p_t\, dY_t^k \quad \text{(Zakai equation)}$$

General approach [Korezlioglu-Mazziotto]

$\triangle$	sample observations (a.s.)
$\triangle\ \triangle$	replace original model with simpler model (weak)
$\triangle\ \triangle\ \triangle$	derive corresponding optimal filter
$\triangle\square$	relate with approximation for the Zakai equation

Sampling observation $\rightarrow$ Splitting-up approximation

$$0 = t_0 < \ldots < t_n < t_{n+1} < \ldots \qquad\qquad \Delta = t_{n+1} - t_n$$

$$\bar{p}^{\Delta}_{n+1} \;=\; \psi_{n+1}\, P^*_{\Delta}\, \bar{p}^{\Delta}_n$$

with

- $\{p^*_t,\ t \geq 0\}$ semigroup associated with L^* provides "explicit" solution of Fokker-Planck equation

$$\frac{\partial}{\partial t} p_t \;=\; L^* p_t$$

- likelihood function

$$\psi_{n+1}(x) \;=\; \exp\Big\{h(x)\,[Y_{t_{n+1}} - Y_{t_n}] - \frac{\Delta}{2}\,|\,h(x)\,|^2\Big\}$$

provides explicit solution of

$$dp_t = \sum_{k=1}^{d} h_k(x)\, p_t\, dY_t^k \qquad \text{(Bayes formula).}$$

Approximate model:

$$Z_n^\Delta = \frac{1}{\Delta}\,[Y_{t_{n+1}} - Y_{t_n}] = \frac{1}{\Delta}\int_{t_n}^{t_{n+1}} Z_s ds$$

$$Z_n^\Delta = h(X_{t_n}) + v_n^\Delta$$

$\overline{p}_n^\Delta$ is exactly the associated optimal filter.

Error estimate:

$$\left\{E^\dagger \, \| p_{t_n} - \overline{p}_n^\Delta \|^2_{L^2}\right\}^{\frac{1}{2}} = O(\Delta)$$

Hint: expand ψ_{n+1}, and $P_\Delta^* = e^{\Delta L^*}$

compare with stochastic Taylor expansion

Remark: intrinsically stable (probabilistic interpretation) [Platen]

Extensions:

- higher-order sampling [FℓG, CDC'89, Tampa]
- correlated noise

  Bensoussan-Glowinski-Rascanu

  Elliott-Glowinski

  Florchinger-LeGland

Further approximation:

$$\begin{aligned} P_\Delta^* &= e^{\Delta L^*} \\ &\simeq (I + \Delta L^*) \quad \text{explicit} \\ &\simeq (I - \Delta L^*)^{-1} \quad \text{implicit} \end{aligned}$$

$$\overline{p}^{\Delta}_{n+1} = \psi_{n+1}(I-\Delta L^*)^{-1}\,\overline{p}^{\Delta}_{n}$$

or:

$$\begin{cases} (I-\Delta L^*)\,\overline{p}^{\Delta}_{n+\frac{1}{2}} = \overline{p}^{\Delta}_{n} \\ \overline{p}^{\Delta}_{n+1} = \psi_{n+1}\,\overline{p}^{\Delta}_{n+\frac{1}{2}} \end{cases}$$

[Clark, '77]

Approximate model:

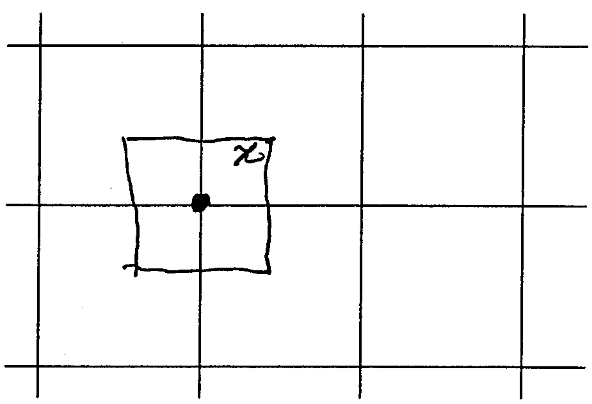
$$\overline{X}^{\Delta}_{n} = X_{\tau^{\Delta}_{n}}$$

$$z^{\Delta}_{n} = h(\overline{X}^{\Delta}_{n}) + v^{\Delta}_{n}$$

$\overline{p}^{\Delta}_{n}$ is exactly the associated optimal filter

still an elliptic PDE to be solved ...

<u>Finite differences</u> [Kushner, Kushner-Dupuis]



grid G_H

For any $x \in G_H$:

$$\frac{\partial u}{\partial x_i}(x) \simeq \begin{cases} \dfrac{u(x+He_i)-u(x)}{H} & \text{if } b_i(x) > 0 \\[2ex] \dfrac{u(x)-u(x-He_i}{H} & \text{if } b_i(x) < 0 \end{cases}$$

$$\frac{\partial^2 u}{\partial x_i^2} \simeq \frac{u(x+He_i) - 2u(x) + u(x-He_i)}{H}$$

$$i \neq j, \ \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \simeq \begin{cases} \cdots & \text{if } a_{ij}(x) > 0 \\ \cdots & \text{if } a_{ij}(x) < 0 \end{cases}$$

$$Lu(x) \simeq \sum_{y \in G_H} L_H(x,y)\, u(y)$$

only neighbors of x actually contribute.

Approximate model:

pure jump Markov process $\{\overline{X}_t^H, \ t \geq 0\}$

taking values in G_H, associated with L_H

$$\overline{X}_n^{H,\Delta} = \overline{X}_{\tau_n^\Delta}^H$$

$$z_n^\Delta = h(\overline{X}_n^{H,\Delta}) + v_n^\Delta$$

Associated optimal filter solves

$$\begin{cases} (I - \Delta L_H^*)\, \overline{p}_{n+\frac{1}{2}}^{\Delta,H} = \overline{p}_n^{\Delta,H} \\ \overline{p}_{n+1}^{\Delta,H} = \psi_{n+1}\, \overline{p}_{n+\frac{1}{2}}^{\Delta,H} \end{cases}$$

Variant: $\psi_{n+1}(x)$ evaluated on G_H

can be very inaccurate when observation noise small. Replace with:

$$R_{n+1}(x) = \max_{x' \in V_H(x)} \psi_{n+1}(x')$$

generalized likelihood ratio.

Convergence:

$$X^{H,\Delta} \Rightarrow X \quad \text{as } \Delta, H \to 0 \text{ independently (implicit).}$$

Hence

$$\langle \bar{p}_n^{\Delta, H}, f \rangle \to \langle p_{t_n}, f \rangle \quad \text{in } L^1 \text{ sense (for } p^t\text{) as } \Delta, H \to 0.$$

General result [Kushner]

$$\text{If } X^n \Rightarrow X, \text{ then } \quad E^n[f(X_t^n) \mid \mathcal{Y}_t] \to E[f(X_t) \mid \mathcal{Y}_t] \quad \text{in } L^1 \text{ sense (for P).}$$

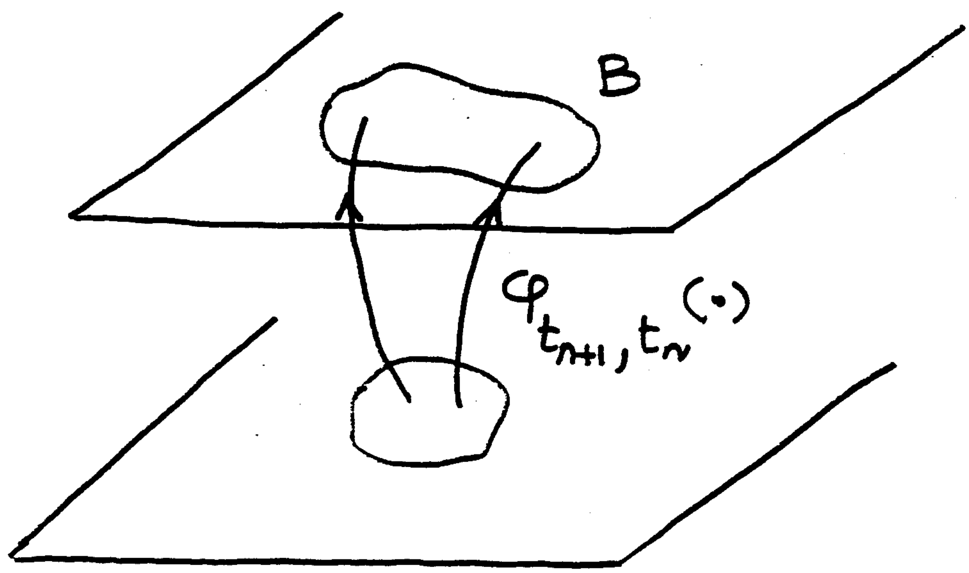
Extensions:

- adaptive local grid refinement [H. Zhang, PhD thesis, 92].

Special case: noise-free state equation

$$\dot{X}_t = b(X_t)$$

$\{\varphi_{t,s},\ 0 \le s \le t\}$ associated flow of diffeomorphisms, i.e., $X_t = \varphi_{t,s}(X_s)$.



$$P[X_{t_{n+1}} \in B] = P[X_{t_n} \in \varphi^{-1}_{t_{n+1}, t_n}(B)]$$

$\{B_0^i,\ i \in I\}$ partition of $\mathbb{R}^m$ at initial time "cells".